

ABOUT THE CAVITY FIELDS IN MEAN FIELD SPIN GLASS MODELS

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We give the explicit expression of the infinite volume limit for the random overlap structures appearing in the mean field spin glass model. These structures have the expected factorization property for the cavity fields, and enjoy invariance with respect to a large class of random transformations. These properties are typical of the Parisi *Ansatz*.

1. Introduction

In previous work [1], [2], by using a simple interpolation argument, we have proven the existence of the thermodynamic limit of the free energy, for mean field disordered models, including the Sherrington-Kirkpatrick model [3], and the Derrida p-spin model [4]. Then, we have extended this argument [5], in order to compare the limiting free energy with the expression given by the Parisi *Ansatz* [6], and including full spontaneous replica symmetry breaking. Our main result is that the quenched average of the free energy is bounded from below by the value given in the Parisi *Ansatz*, uniformly in the size of the system. Moreover, the difference between the two expressions is given in the form of a sum rule, extending our previous work on the comparison between the true free energy and its replica symmetric Sherrington-Kirkpatrick approximation [7]. We have given also a variational bound for the infinite volume limit of the ground state energy per site.

In the meantime, Aizenman, Sims, and Starr [8] have been able to formulate a very attractive extended variational principle, in which the actual value of the free energy, in the infinite volume limit, is expressed through an optimization procedure for which the Parisi ultrametric/hierarchical structures form only a subset of the variational class. This leads the authors to put the question whether “ultrametricity is an inherent structure of the Sherrington-Kirkpatrick mean-field model, or is it only a simplifying assumption”.

Finally, Talagrand [9] has announced a proof, based on a far reaching extension of the methods proposed in [5], of the long expected result that the infinite volume limit of the free energy is precisely given by the Parisi expression.

The main purpose of this paper is to report about the general form of the random overlap structures arising in the infinite volume limit for the cavity fields. We will show that they have a very important factorization property, and, moreover, they are invariant under a large class of random transformations. These properties are typical of the Parisi *Ansatz*.

The organization of the paper is as follows. In Section 2, we will briefly recall the main features, with definitions and results, of the mean field spin glass model. Section 3 contains

the main result of this report. We give only a rapid sketch of the proof, by pointing out the main ideas. A more detailed treatment will be presented elsewhere [10]. Finally, we give also a synthetic outlook for further developments.

2. The basic definitions and results for the mean field spin glass model

The generic configuration of the mean field spin glass model is defined through Ising spin variables $\sigma_i = \pm 1$, attached to each site $i = 1, 2, \dots, N$.

The external quenched disorder is given by the $N(N-1)/2$ independent and identically distributed random variables J_{ij} , defined for each unordered couple of sites. For the sake of simplicity, we assume each J_{ij} to be a centered unit Gaussian with averages $E(J_{ij}) = 0$, $E(J_{ij}^2) = 1$.

The Hamiltonian of the model, in some external field of strength h , is given by the mean field expression

$$H_N(\sigma, h, J) = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i. \quad (1)$$

Here, the first sum extends to all site couples, and the second to all sites.

For a given inverse temperature β , let us now introduce the disorder dependent partition function $Z_N(\beta, h, J)$, the quenched average of the free energy per site $f_N(\beta, h)$, the Boltzmann state ω_N , and the auxiliary function $\alpha_N(\beta, h)$, according to the well known definitions

$$Z_N(\beta, h, J) = \sum_{\sigma_1 \dots \sigma_N} \exp(-\beta H_N(\sigma, h, J)), \quad (2)$$

$$-\beta f_N(\beta, h) = N^{-1} E \log Z_N(\beta, h, J) = \alpha_N(\beta, h), \quad (3)$$

$$\omega_N(A) = Z_N(\beta, h, J)^{-1} \sum_{\sigma_1 \dots \sigma_N} A \exp(-\beta H_N(\sigma, h, J)), \quad (4)$$

where A is a generic function of the σ 's. Notice that the Boltzmann state is depending also on the external noise J_{ij} .

Replicas are introduced by considering a generic number s of independent copies of the system, characterized by the Boltzmann variables $\sigma_i^{(1)}, \sigma_i^{(2)}, \dots$, distributed according to the product state $\Omega_N = \omega_N^{(1)} \omega_N^{(2)} \dots \omega_N^{(s)}$. Here, all $\omega_N^{(\alpha)}$ act on each one $\sigma_i^{(\alpha)}$'s, and are subject to the *same* sample J of the external noise.

The overlap between two replicas a, b is defined according to $q_{ab} = N^{-1} \sum_i \sigma_i^{(a)} \sigma_i^{(b)}$, with the obvious bounds $-1 \leq q_{ab} \leq 1$.

For a generic smooth function F of the overlaps, we define the $\langle \rangle$ averages

$$\langle F(q_{12}, q_{13}, \dots) \rangle = E \Omega_N(F(q_{12}, q_{13}, \dots)), \quad (5)$$

where the Boltzmann averages Ω_N act on the replicated σ variables, and E is the average with respect to the external noise J .

In [1] it was shown how to prove the existence of the limit

$$\lim_{N \rightarrow \infty} \alpha_N(\beta, h) = \sup_N \alpha_N(\beta, h) \equiv \alpha(\beta, h),$$

for all values of the parameters (β, h) .

By using standard convexity arguments, it is easy to prove, for generic values of the parameters (β, h) , *i.e.* by excluding a set of zero measure, that the following limit also exists

$$\lim_{N \rightarrow \infty} \langle q_{12}^2 \rangle_N \equiv \langle q_{12}^2 \rangle,$$

where we have stressed the dependence on N before the limit.

Moreover, the results of [5] show that $\alpha(\beta, h)$ is uniformly bounded from above by the value found in the frame of the Parisi *Ansatz*.

3. The infinite volume limit of the random overlap structure

A very general definition of random overlap structures has been given in [8]. Important references dealing with this concept are [11], [12], [13], [14], [15]. Here we need a much simpler setting, because our random overlap structures will be explicitly constructed through an infinite volume limit.

Let us consider the Boltzmann state, as defined in previous section, called here Ω_N , acting on the N Ising variables called here $\tau_1, \tau_2, \dots, \tau_N$, for the sake of convenience. Let $\eta_j(\tau)$, for $j = 1, 2, \dots$ be τ -conditionally independent Gaussian random variables, with zero mean and variance given by the overlaps

$$E(\eta_j(\tau)\eta_{j'}(\tau')) = \delta_{jj'} q(\tau, \tau'). \quad (6)$$

We introduce also the family of Gaussian random variables, $\kappa(\tau)$, as independent from the η 's, with zero mean and variance given by the square of the overlaps

$$E(\kappa(\tau)\kappa(\tau')) = q^2(\tau, \tau'). \quad (7)$$

We need also the random variables defined by

$$c_j = 2 \cosh \beta(h + \eta_j) = \sum_{\sigma_j} \exp(\beta(h + \eta_j)\sigma_j). \quad (8)$$

Consider also the integer random variable K uniformly distributed on the values $(1, 2, \dots, N)$.

Now we are ready to define our main quantities. For any integer M , and any real λ , introduce

$$E \log \Omega_K(c_1 c_2 \dots c_M \exp(\lambda \kappa)). \quad (9)$$

Here the average E contains the averages over the external noise J_{ij} , appearing in Ω_K , over the variables η and κ , and over K .

It is simple to realize that the previously introduced expression has a very simple interpretation. In fact, we can consider a large number of cavity spins τ coupled with some additional spins σ , and evaluate the change in free energy coming from the M added spins and from a small change in the two-spin coupling in the cavity.

The need to consider the additional average over K is due to the uncomplete control of the corrections to the infinite volume limit for the free energy. This will amount to the consideration of a kind of Cesàro limit in the following. Should the corrections be found of order $O(\frac{1}{N})$, as naturally expected, then a simple limit over the size of the cavity would be sufficient.

The main result of this report is the following.

Theorem 3.1. *For all values of the (β, h) parameters, where the infinite volume limit of the averaged squared overlap $\langle q_{12}^2 \rangle$ is uniquely defined, we have*

$$\lim_{N \rightarrow \infty} E \log \Omega_K(c_1 c_2 \dots c_M \exp(\lambda \kappa)) = M(\alpha(\beta, h) + \frac{\beta^2}{4}(1 - \langle q_{12}^2 \rangle)) + \frac{\lambda^2}{2}(1 - \langle q_{12}^2 \rangle). \quad (10)$$

The proof is long but straightforward. The main ingredients are given by the convergence of the free energy, standard convexity arguments, and the fact that for a very large cavity the added spins σ 's do not interact among themselves, but only with the cavity spins τ through the random fields η . Notice that in the Cesàro limit only the large values of K do really matter in the limit on N . We refer to [10] for all details.

Let us call $E \log \Omega$ symbolically the resulting random overlap limiting structure. Notice that it shares a very important factorization property. In fact, the random variables η, κ become independent under the limiting $E \log \Omega$. As a consequence, it turns out that Ω is invariant under all stochastic transformations of the type

$$\Omega(\cdot) \rightarrow \Omega'(\cdot) \equiv \Omega(c_1 \cdot) / \Omega(c_1),$$

$$\Omega(\cdot) \rightarrow \Omega''(\cdot) \equiv \Omega(\exp(\lambda \kappa) \cdot) / \Omega(\exp(\lambda \kappa)).$$

These factorization and invariance properties are typical of the random overlap structures found in the frame of the Parisi *Ansatz*. It is also clear that in the extended variational principle of Aizenman, Sims and Starr, without loss of generality, we can limit ourselves to the consideration only of factorized overlap structures. In this way we get always bounds uniform in the size of the system.

The set of general factorized overlap structures enjoys a natural convexity property. In fact, if Ω_i , $i = 1, 2$, are factorized, then the new structure, defined by $E \log \Omega_i$, where E contains also a convex average over i , is clearly factorized. This shows that the set of all factorized structures is larger than the Parisi set. In fact, the mixture of two Parisi structures is not Parisi, in general. However, only the extremal structures are relevant in the optimization procedure in the extended variational principle. Therefore, the problem to see whether the optimal structure is ultrametric or not is still open. We plan to dedicate future work to this important problem.

Finally, let us remark that there are strong connections between our general results about limiting random overlap structures, and the stability properties found in the frame of the cavity approach of Mezard-Parisi-Virasoro [16], or the kind of stochastic stability exploited by Aizenman and Contucci in [17].

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